Prime Probability Paradox through Parity, by: Lawrence Xie

Introduction

Prime numbers are integers that only have factors of one and itself. All other integers above two are composite. In many ways, prime numbers are the building blocks for all natural numbers. Although the simple definition of a prime number may be easy to understand, there are still many unanswered questions regarding prime numbers. For instance, is there any formula that can generate consecutive primes? To this day, no such formula has been found. In fact, it requires strong computing power and complex algorithms to efficiently search for new large prime numbers. Another problem associated with prime numbers is the question of how prime numbers are distributed among numbers. The distribution of prime numbers has been a central topic in mathematical studies for over one thousand years.

Another well-known property regarding integers is evenness and oddness, known as parity. A number is even if it is divisible by two; otherwise, it is odd. When considering all natural numbers, it is apparent that every other number is even. All even numbers are composite with the exception of 2, and all prime numbers are odd. However, not all odd numbers are prime.

Purpose

The question of the study is therefore, "If an integer is randomly selected, what is the probability that it is prime?" The tree diagram indicates that there is a relationship between even and oddness (parity) and primality. While the probability of even and odd

numbers is equal, the probability of primes is less than the probability of composites. That is because composite numbers can be even or odd. However, if one can determine the probability of odd composite numbers, the probability of primes can be calculated.
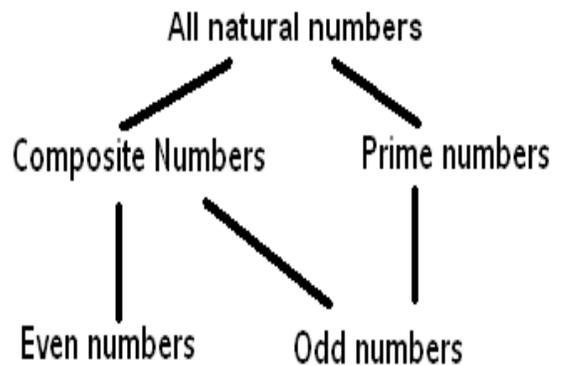
<u>Prelude to the Paradox</u>

An interesting scenario occurs when one examines the probability of prime numbers indirectly through the relationship between parity and primality.

<u>Step 1: Study the parity</u> of composite numbers:

Let a and b be integers

For the product of two evens:

$$2a \times 2b = 2(2ab)$$

For the product of two odds:

$$(2a-1) \times (2b-1) = 2(2ab) - 2(a+b) + 1$$

For the product of one even and one odd:

$$2a \times (2b-1) = 2(2ab-a)$$

- An even number multiplied by an even number always results in an even number.

- An odd number multiplied by an odd number always results in an odd number.

- An even number multiplied by an odd number always results in an even number.

From this the probability of a composite number's parity can be determined[1]. There are three possibilities for each composite number. The factors can be either:

Even x Even → results in even

Even x Odd → results in even

Odd x Odd → results in odd

Two of the possibilities result in a even number. Therefore, the probability of a random composite number being even is 2/3 ≈ 66%

<u>Step 2: Examine the relationship</u> between the probability of an even composite number with the probability of a prime number.

Axiom 1: A randomly chosen integer is 100% likely to be a natural number (by definition).

Axiom 2: All natural integers are either composite (C) or prime (P). Therefore, the probabilities should add to 100%.

Axiom 3: A randomly chosen integer is either even or odd: 50% likely to be odd and 50% likely to be even. Therefore, the probabilities should add to 100%.
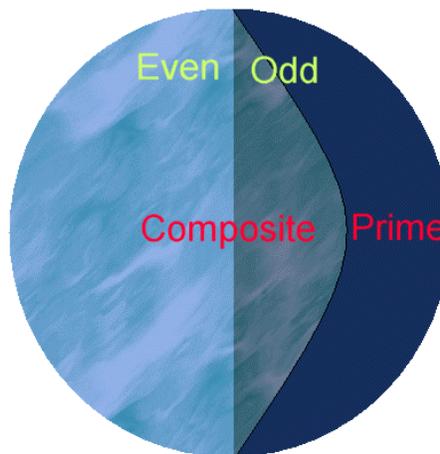
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[1] This assertion is later proved to be false.

All natural integers
1 or 100%

Composite Numbers
C

Prime numbers
P

Even numbers
1/2 or 50%

Odd numbers
1/2 or 50%
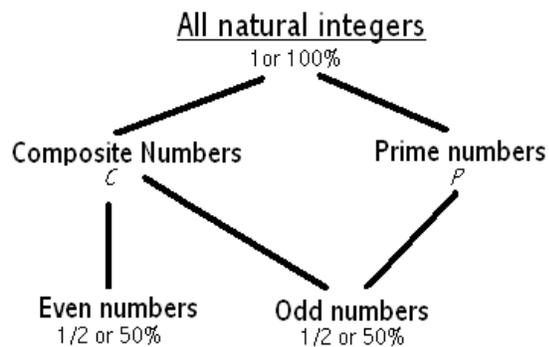
Sum Total: 1 or 100%

Sum Total: 1 or 100%

Sum Total: 1 or 100%

$$\left\{ \begin{array}{l} P(even|composite) \times P(composite) = 1/2 \\ P(composite) + P(prime) = 1 \end{array} \right\}$$

From step 1: $P(even|composite) = 2/3$

$$P(composite) = \frac{1}{2 \times P(even|composite)}$$

$$P(prime) = 1 - \frac{1}{2 \times \frac{2}{3}}$$

$$P(prime) = 1 - \frac{1}{2 \times P(even|composite)}$$

$$P(prime) = 1 - 3/4 = 1/4 = 25\%$$

Thus, from this observation, it appears that a randomly chosen integer is 25% likely to be prime.

The Paradox!:

The reasoning outlined above leads to a conclusion that the probability of an integer being prime is ¼ or 25%. It also suggests that for any given number N, there are approximately N/4 number of primes under N. For example, the calculation predicts that there are 25 primes less than 100, there are 250 primes less than 1000 etc. In order to verify this calculation, the number of primes less than a number x was calculated using a computer program written in Visual Basic. The numerical evidence for the number of primes less than x is shown below.

x	$\pi(x)$	Probability	x	$\pi(x)$	Probability
100	25	25.00%	20 000	2262	11.31%
300	62	20.66%	30 000	3245	10.82%
500	95	19.00%	40 000	4203	10.51%
700	125	17.85%	50 000	5133	10.26%
900	154	17.11%	60 000	6057	10.09%
1100	184	16.72%	70 000	6935	9.90%

Testing this conclusion for integers less than 100 using a computer program yields exactly 25 primes under 100, suggesting that the conclusion is valid. However there are 95 primes less than 500 whereas the earlier conclusion predicted 125 primes! For higher x values, there is an even lower percentage of prime numbers. It is clear that the predicted probability of ¼ goes against numerical evidence and the *prime value theorem*.

Question:

What flaw in reasoning led to the incorrect assertion that randomly chosen integers have a ¼ probability of being prime?

Numerical Evidence and Prime Value Theorem

The prime value theorem states that the number of primes less than or equal to x is approximately $x/(\log x)$.

This means that the probability of a prime number within a number set $\leq N$ is $= \dfrac{\dfrac{N}{\log N}}{N} = \dfrac{1}{\log N}$

In contrast to the probability obtained earlier, the PVT predicts that the probability of a random integer being prime varies according to how large the integer is. With this thought in mind, I was able to determine the flaw in my reasoning earlier.

Paradox Solved:

An error was in Step 1 of the Prelude to the Paradox was found. I incorrectly made the conclusion that a randomly chosen composite number has a 2/3 probability of being even. This flaw in reasoning arose from the assumption that each of the composite number's factors each has a 50% chance of being 2 (Even). In reality, composite numbers are built in accordance with the *fundamental theorem of arithmetic,* which states that every natural number can be written as a unique product of prime numbers. Therefore, a randomly chosen composite number does *not* have a 50% chance of having an even factor. Instead, a composite number's parity depends solely on if at least one of its factors is 2.

Let N be a random composite number

$N = a_1 a_2 a_3 a_4 ...$

N is only even if a_1 is 2. Since the number possible prime numbers factors goes up for increasing N values, the probability that one of these factors is 2 decreases. As a result:

$N\uparrow$, $evenComposite\downarrow$, $\therefore$ $prime\downarrow$

With this adjustment to our earlier conclusion, it is possible to explain the decreasing prime density through increasing number of prime number factors. Unfortunately, this conclusion also implies that it is impossible to find a way to predict prime probability through parity of composite numbers. This is because it is necessary to predict the prime factor buildup for composite numbers in order to evaluate parity. However doing so would require knowledge of prime probability in the first place!

<u>Reasoning Reworked</u>

Although the original task of finding a way to predict prime probability through parity of composite numbers was unsuccessful, there was still useful knowledge gained through the reasoning. I decided to rework the probability calculations for a more general formula. The property of being even or odd is actually just a test of whether a number is divisible by 2. I reasoned that my initial calculation should work for all numbers:

Axiom: The probability of N being divisible by n is $1/n$

$$\left. \begin{cases} P(ndivisible|composite) \times P(composite) = 1/n \\ P(composite) + P(prime) = 1 \end{cases} \right\}$$

$$P(composite) = \frac{1}{n \times P(ndivisible|composite)}$$

$$\frac{1}{n \times P(ndivisible|composite)} + P(prime) = 1$$

$$P(ndivisible|composite) = \frac{1}{n \times (1 - P(prime))}$$

$\therefore$ A random composite integer has

a $\dfrac{1}{n \times (1 - P(prime))}$ chance of being

divisible by n.

<u>Conclusion</u>

Although this study does not appear to give any direct applications, it provides a unique way to understand prime number distribution through probability. Through the process of analyzing prime number data, one can see how the problem of distribution of prime numbers spreads widely throughout probability and understandings of parity and divisibility.

Acknowledgements

I would like to thank Ms. Kuz for helping me write the program to calculate the number of primes less than a given value. I would also like to thank my father for tiding my abstract to ensure that is clear and logical; also for helping me with the visual depiction of the natural number/even odd/prime composite sets.

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